MATH 5061 Problem Set 1[1]
Due date: Jan 22, 2020

Problems: (Please hand in your assignments to me after class. Those questions marked with a † are optional.)

1. † Prove that any two smooth structures on the topological manifold $\mathbb{R}$ are diffeomorphic. *(Hint: Let M be $\mathbb{R}$ equipped with a smooth structure and the real line with the standard smooth structure by $\mathbb{R}$. It suffices to find a smooth function $f : M \to \mathbb{R}$ such that $f_* = df$ is non-zero at every point of M. Show that it is the same as finding a no-where vanishing 1-form ω on M. Prove that such 1-forms exist locally and use partition of unity to extend to all of M.)*

2. (a) Prove that $(M_1 \times M_2) \times M_3$ is diffeomorphic to $M_1 \times (M_2 \times M_3)$, and that $M_1 \times M_2$ is diffeomorphic to $M_2 \times M_1$.

   (b) Prove that a map $f : M \to M_1 \times M_2$ is smooth if and only if both of the maps $\pi_1 \circ f$ and $\pi_2 \circ f$ are smooth where π_i denotes the projection map from $M_1 \times M_2$ to M_i.

3. A function f on a manifold M^n has a *critical point* at $p \in M$ if $Xf(p) = 0$ for all $X \in T_pM$. The Hessian matrix of f at p is then defined by $H(X,Y) = XYf$ for $X, Y \in T_pM$.

   (a) Note that the definition requires that Y be extended as a vector field to a neighborhood of p. Show that $H(X,Y)$ does not depend on the extension of Y.

   (b) Show that $H(X,Y) = H(Y,X)$.

   (c) Show that there is a basis $e_1, \cdots, e_n$ for T_pM such that T_pM such that $H(e_i, e_j) = \lambda_i \delta_{ij}$ where $\lambda_i \in \{-1, 0, 1\}$. Show that the number of 1's, -1's, and 0's is independent of the choice of basis.

4. If a manifold has a distribution, we say that a curve is *horizontal* if it is tangent to the distribution at each point.

   (a) If a distribution is integrable, show that the set of points which can be joined by a horizontal curve to a given point p is contained in the integral submanifold through p.

   (b) Consider the two dimensional distribution in $\mathbb{R}^3$ spanned by the independent vector fields $X = \partial/\partial x$ and $Y = \partial/\partial y + x\partial/\partial z$. Show that any two points in $\mathbb{R}^3$ can be joined by a horizontal curve. *(Hint: consider the projection to the xy-plane and find an interpretation of z for this projected curve. It is simplest to think of the distribution as being the vectors that are annihilated by the 1-form $dz - xdy$ so that this form evaluates to zero along a horizontal curve.)*

5. Let $F : M \to N$ be a smooth map.

   (a) Assume that X_1, X_2 are smooth vector fields on M and Y_1, Y_2 are smooth vector fields on N such that $F_*(X_i) = Y_i$ for $i = 1, 2$. Show that $F_*([X_1, X_2]) = [Y_1, Y_2]$.

   (b) Assume that F is a $1-1$ immersion, and let $\Sigma = F(M)$. Suppose that Y_1, Y_2 are vector fields defined in a neighborhood of Σ and are tangential to Σ at each point of Σ. Show that $[Y_1, Y_2]$ is tangential at each point of Σ. Show further that if Z_1, Z_2 are vector fields with $Z_i = Y_i$ on Σ, then $[Y_1, Y_2] = [Z_1, Z_2]$ on Σ.

6. Let V^m be a real vector space. A non-degenerate symmetric bilinear form g on V is called a *scalar product.*

   (a) Show that for any scalar product, there is an "orthonormal" basis; that is, a basis $e_1, \cdots, e_m$ with $g(e_i, e_j) = \lambda_i \delta_{ij}$ where λ_i is 1 or -1. Show that the number ν of -1's which occur is independent of the orthonormal basis chosen.

   (b) The set of vectors satisfying $g(v,v) = 0$ is called the null cone. Show that the largest dimensional subspace contained in the null cone has dimension equal to $\min\{\nu, m - \nu\}$.

[1]Last revised on January 20, 2020

(c) Show that a scalar product defines a natural isomorphism I of V with its dual space V^*. If $e_1, \cdots, e_m$ is a basis for V, and $a^1, \cdots, a^m$ are the coordinates of a vector v with respect to this basis, find the coordinates of $I(v)$ with respect to the corresponding dual basis.

7. Let V be a real vector space with a scalar product g which is neither positive nor negative definite. Let b be any other scalar product on V. Show that the following four conditions are equivalent:

   (i) $b = cg$ for some $c \in \mathbb{R}$,

   (ii) $b(v, v) = 0$ for every null vector v,

   (iii) $|b(v, v)|$ is bounded on $\{v : g(v, v) = -1\}$,

   (iv) $|b(v, v)|$ is bounded on $\{v : g(v, v) = 1\}$.